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Simultaneous Optimization of Sailplane Design and Its Flight Trajectory

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An aircraft design and its flight trajectory problem should be formulated simultaneously when an exact flight mission is determined or an extremely high-performance level is required. This article presents a block diagonal sequential quadratic programming approach by introducing interface variables between the two problems to efficiently solve simultaneous optimization problems. As numerical examples, the design of a sailplane for a flight distance competition is studied. The need of simultaneous optimization will be demonstrated by comparing a conventional design process; i.e., a lift–drag ratio maximization approach.

Introduction

IN an aircraft design phase, a configuration and geometrical parameters of an aircraft are optimized for assumed or prescribed flight paths. After an aircraft has been designed, actual flight paths or trajectories are optimized in consideration of fuel efficiency or flight time.

Numerical optimization techniques have been developed for both aircraft design and trajectory problems. Aircraft design optimization problems have been successfully solved by using a mathematical programming formulation. Vanderplaats[1] optimized a supersonic aircraft design for a prescribed mission profile. Grossman et al.[2] studied a sailplane wing design for a simplified mission performance. Dovi and Wrenn[3] solved a wing-body transport design for an assumed mission profile. In these studies, design parameters, such as wing area, aspect ratio, sweep-back angle, and thickness-to-chord ratio were selected as the design variables to be optimized. On the other hand, flight trajectory problems can be formulated as optimal control problems[4] that have been solved by using numerous solution approaches, e.g., calculus of variations, maximum principle, and dynamic programming. Recent investigations have revealed the advantages of a mathematical programming approach for optimal control problems.[5–7] In a mathematical programming formulation, control variables are discretized as the design variables to be optimized, and a performance index is maximized while satisfying boundary conditions and path constraints. It follows from this that mathematical programming algorithms handle inequality constraints in an efficient manner, and do not have as excessive computer-storage requirements as dynamic programming. Furthermore, a mathematical programming formulation can combine design problems with flight trajectory problems. While it is seldom that these two problems in different disciplines are solved simultaneously, Martens[8] presented a jet transport sizing problem including mission analysis.

An increase in computer capabilities and development of optimization algorithms make it possible to solve multidisciplinary optimization for practical problems. Since multidisciplinary optimization problems must deal with multiple analysis programs in different disciplines and a huge number of design variables, several solution methods have been proposed. Sobieski and others[9,10] have developed a decomposition approach based on sensitivity analysis. This article applies a sequential quadratic programming (SQP) method[11] as a mathematical programming formulation, and proposes a block diagonal approach by introducing interface variables to efficiently solve the multidisciplinary optimization problems.

Following the guidelines of the Japanese sailplane competition,[12] sailplane design problems are investigated. In the competition, a sailplane that is controlled by a human pilot descends from a height of 10 m, and a flight distance is measured at touchdown. Except that a runway is limited within 10 m, there are no restrictions on its size and weight. The design process for this sailplane has two interrelated elements: 1) an aircraft design phase and 2) a flight control phase. Both problems must be solved simultaneously to obtain a maximum flight distance. Therefore, this article presents a simultaneous optimization method for a sailplane design and its flight trajectory. Additionally, the presented design method is compared with a conventional design method in which a sailplane is designed to maximize a lift–drag ratio (L/D) without consideration of the trajectory optimization.

SQP Optimization Problem

A mathematical programming problem is generally formulated as

$$\text{Minimize:} \quad J(\boldsymbol{d}) \tag{1}$$

$$\text{Subject to:} \quad c_i(\boldsymbol{d}) = 0, \qquad i \in E \tag{2}$$

$$c_i(\boldsymbol{d}) \geq 0, \qquad i \in K \tag{3}$$

where $\boldsymbol{d}$ and J are the design variables vector and an objective function, respectively, and c_i denotes equal constraints E or unequal constraints K.

An SQP method finds the optimal solutions by solving the following quadratic programming problem in a sequential manner[11]:

$$\text{Minimize:} \quad \nabla_d J(\boldsymbol{d}^k)\boldsymbol{e} + \tfrac{1}{2}\boldsymbol{e}^T\boldsymbol{B}^k\boldsymbol{e} \tag{4}$$

$$\text{Subject to:} \quad c_i(\boldsymbol{d}^k) + \nabla_d c_i(\boldsymbol{d}^k)\boldsymbol{e} = 0, \qquad i \in E \tag{5}$$

$$c_i(\boldsymbol{d}^k) + \nabla_d c_i(\boldsymbol{d}^k)\boldsymbol{e} \geq 0, \qquad i \in K \tag{6}$$

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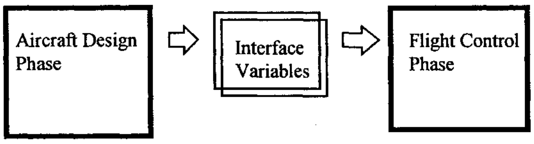

Fig. 1 Interface variables between two phases.

where d^k is the vector d at the kth iteration step, ∇_d denotes the gradient of a function associated with the design variables d, and e is the update vector $(=d^{k+1} - d^k)$. Matrix B in Eq. (4) denotes the approximation of the Hessian matrix H for the following Lagrange function $\mathcal{L}$:

$$\mathcal{L}(d, \lambda) = J(d) - \lambda^T \begin{bmatrix} c_{i\in E} \\ c_{i\in K} \end{bmatrix} \tag{7}$$

$$H = \nabla_{dd}\mathcal{L} \tag{8}$$

where λ is the Lagrange variables vector. The SQP formulation assumes the initial B to be the identity matrix and modifies the matrix B through iteration steps by using gradient information. This article uses the Broyden–Fletcher–Goldfarb–Shanno (BFGS) formulation[13] to calculate the B matrix.

Block SQP for Simultaneous Optimization

Mission performance will be maximized through both aircraft design optimization and trajectory optimization. Hence, the design variables consist of both aircraft design parameters d_D and control parameters d_C.

The objective of the integrated problem is written as

$$\text{Minimize:} \quad J(d_D, d_C) \tag{9}$$

where the objective function J is related to the mission performance. Constraints in both problems are given as

$$\text{Subject to:} \quad g_D(d_D) \geq 0 \tag{10}$$

$$\text{Subject to:} \quad g_C(d_D, d_C) \geq 0 \tag{11}$$

where g_D represent design constraints for structural characteristics, flight stability, and aircraft size limitations, and g_C denote constraints in the flight control problem, e.g., terminal conditions, path, and control constraints (see Appendix A).

To solve this problem efficiently, interface variables d_I are introduced between the aircraft design phase and the flight control phase. The interface variables represent aircraft characteristic parameters appearing in the flight control problems, such as weight, wing loading, and aspect ratio. Since the interface variables are the functions of aircraft design parameters d_D (Fig. 1), the problem can be formulated as

$$\text{Minimize:} \quad J(d_I, d_C) \tag{12}$$

$$\text{Subject to:} \quad g_D(d_D) \geq 0 \tag{13}$$

$$d_I = d_I(d_D) \tag{14}$$

$$g_C(d_I, d_C) \geq 0 \tag{15}$$

Since J and g_C do not have d_D explicitly, it is not necessary to calculate the gradients of those functions associated with d_D. Furthermore, the B matrix in Eq. (4) can be reconstructed in the following block diagonal form:

$$B = \begin{bmatrix} B_{DD} & 0 & 0 \\ 0 & B_{II} & B_{CI}^T \\ 0 & B_{CI} & B_{CC} \end{bmatrix} \tag{16}$$

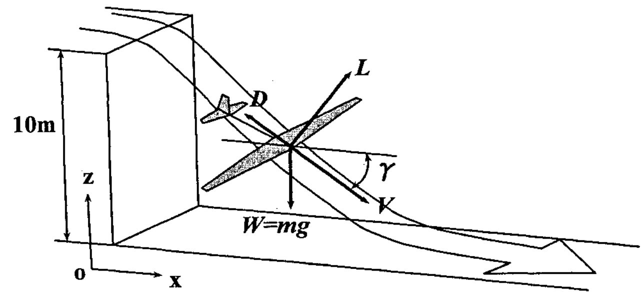

Fig. 2 Flight trajectory in flight distance competition.

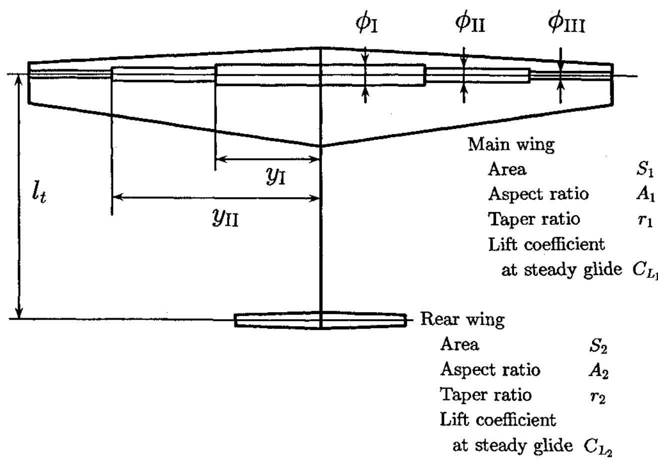

Fig. 3 Design variables of sailplane.

In the SQP formulation, the matrix B is updated in each iteration step.[13] The diagonal characteristics of B can save both the computing time and computer storage by omitting the null parts in B, and increase the numerical accuracy.

Sailplane Design Examples

Aircraft Design Phase

The design problem of a sailplane for the Japanese sailplane competition (Fig. 2) is considered. Although several configurations are available, a conventional glider is assumed as shown in Fig. 3. A sailplane has a straight wing with a single pipe spar.[14] EPPLER748 (Ref. 15) was selected as its wing section, and a structure frame was made from an aluminum tube. The wing spar was divided into three sections so as to select an optimal tube's diameter (Fig. 3). Design variables are listed in Table 1.

The design constraints in the aircraft design phase are summarized in the following way:

Geometrical Size

Although there are no restrictions on size and weight in the regulations, the following constraints are introduced so that a pilot can run up while holding a sailplane:

Weight W:

$$W \leq 35, \text{ kg} \tag{17}$$

Total length l_{total}:

$$l_{\text{total}} \leq 5, \text{ m} \tag{18}$$

Additionally, a spar diameter at wingtip ϕ_{III} must be less than wingtip thickness t_{tip}:

$$\phi_{III} \leq t_{\text{tip}} \tag{19}$$

Table 1 Design variables in aircraft design phase

Design variables	Symbol
Main/rear wings	
Areas	S_1, S_2
Aspect ratios	A_1, A_2
Taper ratios	r_1, r_2
Lift coefficients at steady glide	C_{L_1}, C_{L_2}
Distance of two wings	l_t
Main wing	
Spar divide positions	y_I, y_{II}
Spar diameters	$\phi_I, \phi_{II}, \phi_{III}$
Flight conditions	
Maximum load factor	$n_{\max}$
Steady glide speed	V_c

Structural Characteristics

Maximum stress of spar and fuselage pipes must be less than the final stress limit σ_{llm} at $1.5 \times n_{\max}$ load condition:

$$\sigma \leq \sigma_{\text{llm}} \tag{20}$$

It is assumed that the rear wing's span and area are constrained as

$$b_2 \leq 3, \text{ m}, \qquad S_2 \geq 1, \text{ m}^2 \tag{21}$$

Flight Condition

To keep the controllability of a sailplane, the steady glide speed V_c and lift coefficients at steady glide are restricted as

$$V_c \leq 15, \text{ m/s}, \qquad C_{L_{1,2}} \leq C_{L_{1,2}}^{\max} \tag{22}$$

Although a point mass model is used in the dynamic equations (Appendix B), trim and longitudinal stability conditions must be considered to determine the tail volume.

To formulate the previous design constraints, the following mathematical models are employed:

Aerodynamic characteristics: The lift and the induced drag are calculated by using the vortex-line theory.[16] The wing profile drag is estimated from two-dimensional experimental data of the given airfoil. The parasite drag coefficient of a fuselage is assumed to be 0.06.

Structure characteristics: The stress of each aluminum tube is calculated by using a beam theory in which load distribution was found in the aerodynamic calculations without taking into consideration aeroelastic effects. The diameter and the thickness of a tube are assumed as (20 and 0.8 mm) for a rear wing, and (60 and 1.2 mm) for a fuselage. The diameters of a main wing spar are selected as the design variables, and the thickness of each tube is assumed to be 2% of the tube diameter.

Longitudinal stability: A sailplane is controlled by shifts of the pilot's weight position. The trim condition and the longitudinal stability conditions are given by using the aerodynamic and inertia characteristics.

Flight Control Phase

The optimal flight strategy is determined by solving optimal control problems for a point mass model of an aircraft. The design variables in the flight control phase are lift coefficients and the flight time t_f. Note that the lift coefficient as a time function is discretized by dividing a time scale into small elements and by approximating the variables at each element (Appendix A). In the numerical optimization as shown in Appendix A, the initial value of t_f is assumed and the time scale is divided into 40 elements with assumed lift coefficients. The time step size was determined after the investigation on the sensitivity of the step size to the results.

Initial conditions of the dynamic equations are given as

$$x(0) = 0, \text{ m} \tag{23}$$

$$z(0) = 10, \text{ m} \tag{24}$$

$$\gamma(0) = -3, \text{ deg} \tag{25}$$

$$V(0) = \sqrt{2000/(W + 60)}, \text{ m/s} \tag{26}$$

where the pilot weight is 60 kg, and the initial velocity $V(0)$ is estimated by assuming that the plane gains the same kinetic energy on the 10-m runway. By integrating the dynamic equations in Appendix B, the following constraints can be formulated:

Terminal condition:

$$z(t_f) = 0, \qquad \gamma(t_f) = 0 \tag{27}$$

Path constraint: The flight height is greater than 0 and load factor is less than $n_{\max}$ in each time element:

$$z \geq 0, \qquad n \leq n_{\max} \tag{28}$$

Control constraint: A lift coefficient is less than the stall limit:

$$C_L \leq C_L^{\max} \tag{29}$$

Optimization

To demonstrate the need of simultaneous optimization, a maximum lift–drag ratio design is also considered. If a sailplane flies downward at a steady glide speed, the flight distance is proportional to the *L/D* of a plane. The two optimization designs are formulated as follows.

Simultaneous Design

The design variables both in the aircraft design phase and in the flight control phase are optimized to obtain the maximum flight distance. Table 2 shows the interface variables $\boldsymbol{d}_I$. The design object is to maximize the flight distance $x_f = x(t_f)$; thus, the objective function is given as

$$\text{Minimize:} \quad J = -x_f(\boldsymbol{d}_I, \boldsymbol{d}_C) \tag{30}$$

The design constraints are the unequal and equal constraints both in the airplane design and the flight control phase. Note that equal constraints are introduced to define the interface variables as shown in Eq. (14).

L/D Maximum Design

In *L/D* maximization design, the design phase and the flight control phase are separated in the following manner: the objective function in the design phase is the lift–drag ratio of a plane at the steady glide condition, and the objective function in the control phase is the flight distance of the plane obtained in the design phase. Note that the maximum load factor $n_{\max}$ and the steady glide speed V_c must be specified in the aircraft design phase. The values of $n_{\max}$ and V_c were selected as those obtained in the simultaneous design.

Table 2 Interface variables

Interface variables	Symbol
Zero lift drag coefficient	C_{D_0}
Airplane efficiency × aspect ratio	εA_1
Vehicle weight	W
Wing area	S_1
Maximum load factor	$n_{\max}$

Table 3 Characteristics of optimized sailplanes

Characteristics	Simultaneous opt.	*L/D* maximum
S_1, m^2	14.6	9.42
W, kg	35.0	29.1
A_1	36.4	40.0
W/S_1, kg/m^2	6.52	9.46
L/D	29.0	33.6
x_f, m	396.7	330.6

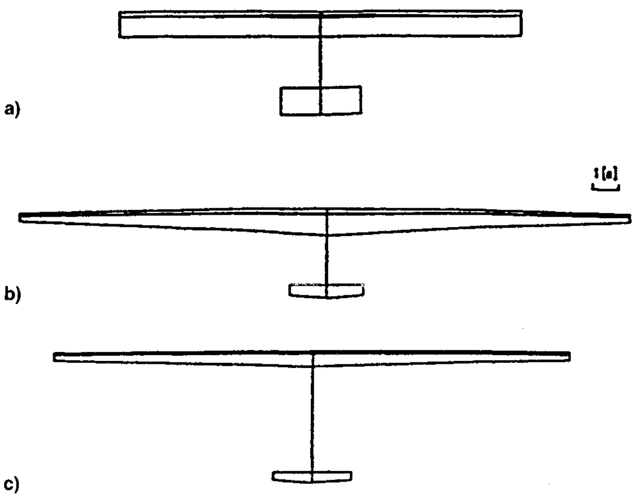

Fig. 4 Initial and optimized shapes: a) initial design, b) simultaneous design, and c) *L/D* maximum design.

Numerical Results

Iteration process in SQP refines the initial design in Fig. 4a to maximize the flight distance in the simultaneous design or to maximize the lift–drag ratio in the *L/D* maximum design until all design variables converge. Figures 4b and 4c illustrate the sailplane shapes obtained by the simultaneous design and the *L/D* maximum design, respectively. The characteristic data of the two planes are compared in Table 3.

Table 3 indicates that the flight distance of the simultaneous design is longer than that of the *L/D* maximum design. This result and the difference of the obtained shapes in Figs. 4b and 4c are caused by the following design policies in the two methods:

1) The *L/D* maximum design extremely increases wing aspect ratio to reduce the induced drag.

2) The simultaneous design reduces the wing loading by increasing wing area within the weight limit, while the *L/D* is sacrificed. Note that if the 35-kg weight limit is raised, better performance can be obtained in the simultaneous design. This weight limit is somewhat arbitrary since there are no regulations in the rules. However, it must be introduced so that the pilot can hold and run up on the 10-m runway as mentioned in the Aircraft Design Phase section.

Figure 5 shows the flight paths, the control variables, and the flight speeds of the optimized sailplanes. The case of interest here is that the optimal flight trajectories of the two sailplanes are different. The sailplane designed by *L/D* maximization dives quickly and flies near the ground because the ground effect can reduce the induced drag (see Appendix B). On the other hand, the sailplane designed in simultaneous optimization flies at steady glide in the early flight stage, and then flies near the ground. It is supposed that although the *L/D* maximum design assumed the steady glide condition, the sailplane cannot generate the steady glide because of the higher wing loading. On the other hand, the sailplane designed in simultaneous optimization can generate the steady flight part with the low glide speed. The low speed can decrease the aerodynamic drag. Furthermore, Fig. 5c indicates that the simultaneous design has a lower touchdown or stall speed than the *L/D* maximum design. Those are because the flight distance of the simultaneous design is longer than that of the *L/D* maximum design.

Finally, the effectiveness of the proposed block diagonal SQP algorithm is confirmed in numerical calculations. The numerical examples in this study can be solved directly without using the interface variables. The number of iterations and the CPU time for both cases are compared in Table 4. This indicates that the block diagonal approach can save both the iteration number and the computational time.

Table 4 Iteration number and CPU time

	SQP	Block diagonal SQP
Number of iterations	52	43
CPU time normalized with SQP case	1.0	0.59

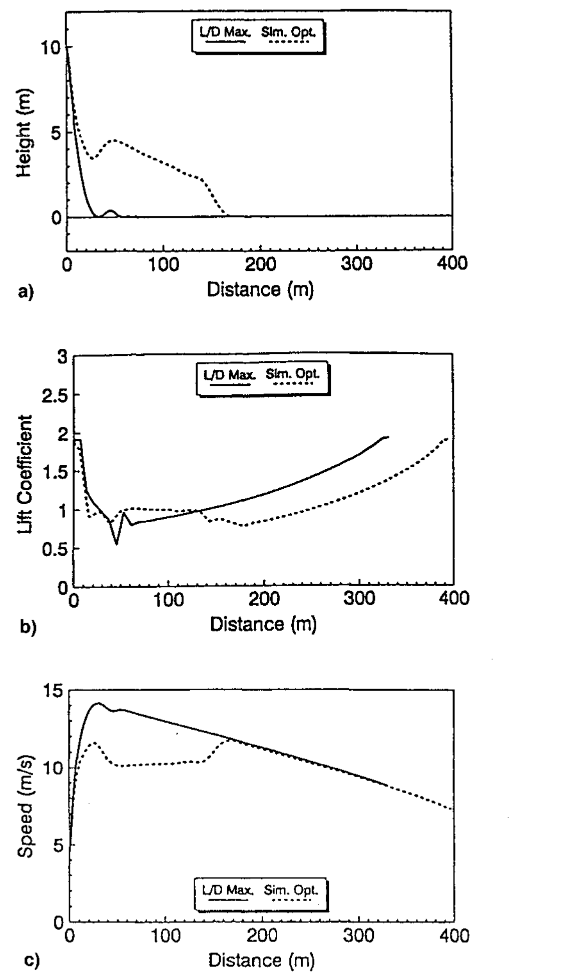

Fig. 5 Optimal flight solutions: a) flight trajectory, b) lift coefficient, and c) flight speed.

Conclusions

The design process of an airplane and the flight trajectory problem were integrated in a simultaneous optimization formulation. A huge number of design problems must be dealt with to solve simultaneous optimization problems. This article proposed the introduction of interface variables to obtain the diagonal form of the approximate Hessian matrix in sequential quadratic programming. Sailplane design for the flight distance competition was considered for numerical examples. The need of the simultaneous optimization has been demonstrated by comparing the present design with the conventional maximum lift–drag ratio design. Finally, it was confirmed that the proposed block SQP algorithm was effective in numerical calculations.

Appendix A: Mathematical Optimization Approach for Optimal Control Problem

The dynamic equations can be expressed in the following state equations:

$$\frac{\mathrm{d}\boldsymbol{x}_s(t)}{\mathrm{d}t} = \dot{\boldsymbol{x}}_s(t) = \boldsymbol{f}[\boldsymbol{x}_s(t), \boldsymbol{u}(t), t] \tag{A1}$$

where $\boldsymbol{x}_s(t)$ and $\boldsymbol{u}(t)$ are the vectors of states and control variables, respectively. At the initial time $t = 0$, and the final or terminal time t_f, some states may be specified as

$$\boldsymbol{\psi}_{\text{ini}}[\boldsymbol{x}_s(0)] = 0, \qquad \boldsymbol{o\psi}_f[\boldsymbol{x}_s(t_f)] = 0 \tag{A2}$$

The upper or lower limits of the state or control variables may be determined during a flight path ($0 \leq t \leq t_f$). This type of constraint is called a path constraint and is defined as

$$\boldsymbol{\nu}^{\text{lower}}(t) \leq \boldsymbol{\nu}[\boldsymbol{x}_s(t), \boldsymbol{u}(t)] \leq \boldsymbol{\nu}^{\text{upper}}(t) \tag{A3}$$

The control variables $\boldsymbol{u}(t)$ are sought to minimize or maximize a given performance index

$$J = J[\boldsymbol{x}_s(t), \boldsymbol{u}(t), t] \tag{A4}$$

To transform an optimal control problem to a parameter optimization problem, the control variables are approximated by defining discrete parameters $\boldsymbol{u}_p$. When a set of time functions $\boldsymbol{\theta}$ is introduced, the control variables are represented as

$$\boldsymbol{u}(t) \approx \boldsymbol{\theta}_1(t)\boldsymbol{u}_1 + \boldsymbol{\theta}_2(t)\boldsymbol{u}_2 + \cdots + \boldsymbol{\theta}_N(t)\boldsymbol{u}_N = \Theta(t)\boldsymbol{u}_p \tag{A5}$$

$$\boldsymbol{u}_p^T = (\boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{u}_N^T) \tag{A6}$$

$$\Theta(t) = [\boldsymbol{\theta}_1(t), \boldsymbol{\theta}_2(t), \ldots, \boldsymbol{\theta}_N(t)] \tag{A7}$$

While any kind of approximations can be utilized, this article employs a piecewise linear approximation; hence, the time scale is divided into small elements and a control variable in each time element is approximated as a linear function of t.

When the final time is not specified, both t_f and u_p are considered as the design variables to be optimized. By integrating the state equations with assumed design variables, the terminal conditions and the path constraints are represented as the functions of the design variables. As a result, the problem can be formulated as a mathematical programming problem.[7]

Appendix B: Dynamic Equations of a Sailplane

The dynamic equations of a sailplane as a point mass m are written as

$$\dot{V} = -(1/2m)\rho V^2 S_1 C_D - g \sin\gamma \tag{B1}$$

$$\dot{\gamma} = (1/2m)\rho V S_1 C_L - (g/V)\cos\gamma \tag{B2}$$

$$\dot{x} = V\cos\gamma \tag{B3}$$

$$\dot{z} = V\sin\gamma \tag{B4}$$

where V, γ, and (x, z) are the flight speed, the path angle, and the position coordinates, respectively; and C_L, ρ, and g are the lift coefficient, the air density, and the gravity acceleration, respectively. The drag coefficient C_D is represented as

$$C_D = C_{D_0} + (C_L^2/\pi\varepsilon A_1^*) \tag{B5}$$

where C_{D_0} and ε are the zero-lift drag coefficient and the airplane efficiency, respectively; and A_1^* is the main wing aspect ratio modified because of the ground effect in the following manner[17]:

$$A_1^* = A_1 \frac{33(z_1/b_1)^{3/2} + 1}{33(z_1/b_1)^{3/2}} \tag{B6}$$

where z_1 and b_1 are the main-wing height from the ground and the main-wing span. This modification indicates that the induced drag decreases when a wing approaches the ground.

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